Unitary Entanglement Generation in Hierarchical Networks

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Abstract

The construction of large-scale quantum computers will require modular architectures that allow physical resources to be localized in easy-to-manage packages. In this work, we examine the impact of different graph structures on the preparation of entangled states. We first explain the hierarchical product framework in which modular graphs can be easily constructed. We argue that the hierarchies constructed thus have favorable properties for quantum information processing, such as a small diameter and small total edge weight. We present numerical and analytical results on the speed at which large entangled states can be created on nearest-neighbor grids and hierarchy graphs, and show that suitably designed hierarchies can perform favorably in comparison to grid architectures.

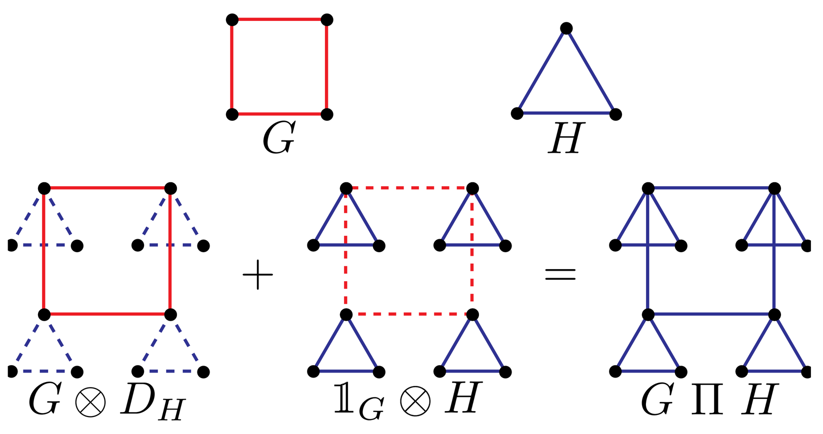
Background

A *graph* $G = (V, E)$ is given by a set of vertices V, and a set of edges between the vertices E, where an edge between distinct vertices i and j is denoted e_{ij}. We consider *weighted* graphs, where we assign a weight $w_{ij} \in \mathbb{R}$ to each pair of vertices $(i, j) \in V \times V$. Two vertices i and j are said to be disconnected if $w_{ij} = 0$, and connected by an edge with weight $w_{ij} \neq 0$ otherwise. Self-edges are disallowed.
Alternatively, we may describe graphs via the *adjacency matrix*, whose rows and columns are labeled by the vertices in V, and whose entries hold edge weights. A closely related matrix is the *algebraic Laplacian* L, which is especially useful in the study of connectivity properties and graph dynamics. The adjacency matrix A and algebraic Laplacian L_G for a graph G are given by

$$A_{ij} = \begin{cases} 0, & \text{if } i = j, \\ w_{ij}, & \text{if } i \neq j, \end{cases} \quad L_{ij} = \begin{cases} v_i, & \text{if } i = j, \\ -w_{ij}, & \text{if } i \neq j, \end{cases}$$

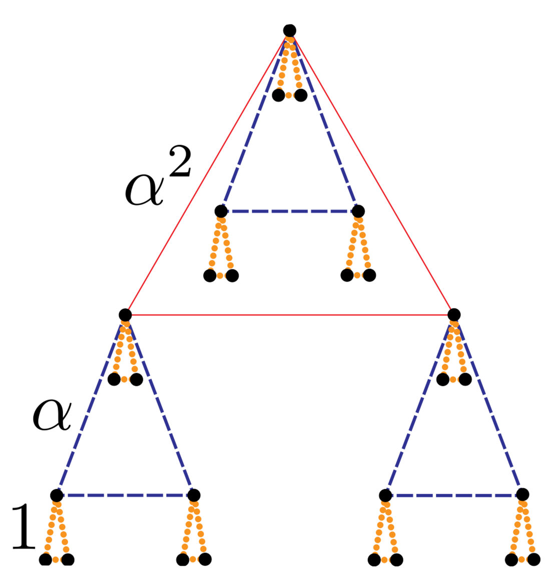
Hierarchical Product

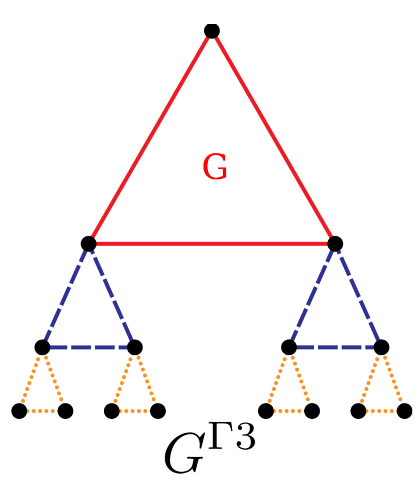
Given graphs G and H, the *hierarchical product* $G \sqcap H$ is a graph with one copy of graph G, with a copy of H attached to each node of G (see figure below). We call the copy of G as the upper level, and the copies of H the lower level of the hierarchy $G \sqcap H$.



$$\mathbb{1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix}.$$

Generalizations to the above formalism allow greater flexibility. First, we could let the levels of the hierarchy be *weighted* (figure below, left). This captures the notion that links scale with the level of the hierarchy in some parameter, which could be, e.g., wire length, or communication bandwidth. In classical architectures, the notion of *fat trees* is closely related.
We may also allow *truncation*, i.e., where we attach copies to only some of the nodes of the upper level (figure below, right). This way, one could equip modules with *communicator qubits* that serve to connect modules across levels, but do not themselves support further sub-hierarchies.
Both generalizations can be made via minor modifications to the equation relating the hierarchy Laplacian to its constituent graph Laplacians (see figure above).





Graph Connectivity Comparisons

As an easy comparison, we tabulate the diameter, weighted diameter, and the max. degree of various topologies commonly seen in quantum architectures.

Graph	δ_w	Δ	w
Complete graph K_N	const.	N	N^2
⋆ Star S_N	const.	N	N
Cycle C_N	N	const.	N
Square grid	$N^{1/d}$	const.	N
⋆ Weighted hierarchy $K_n^{\Gamma_\alpha i}$	$N^{\log_n \alpha}$	const.	N
⋆ Unweighted hierarchy $K_n^{\Gamma i}$	$\log_n N$	const.	N

Table: The scaling with N of three key graph parameters: weighted diameter δ_w, maximum degree Δ, and edge weight w. A star has been placed next to the two graphs we find to be Pareto efficient.

Secondly, the Laplacian spectra of graphs provide key information on connectivity. In particular, the second-smallest eigenvalue λ_2 (also known as the *algebraic connectivity*) bounds other connectivity measures such as diameter, mean distance between nodes, and Cheeger constant, and may therefore be a good measure of connectivity. Larger λ_2 implies "more connected".
The spectra of hierarchy Laplacians can be computed efficiently, due to the algebraic relation

$$\operatorname{spec}(K \sqcap L) = \bigsqcup_{j=1}^{|K|} \operatorname{spec}(\alpha \kappa_j D + L).$$

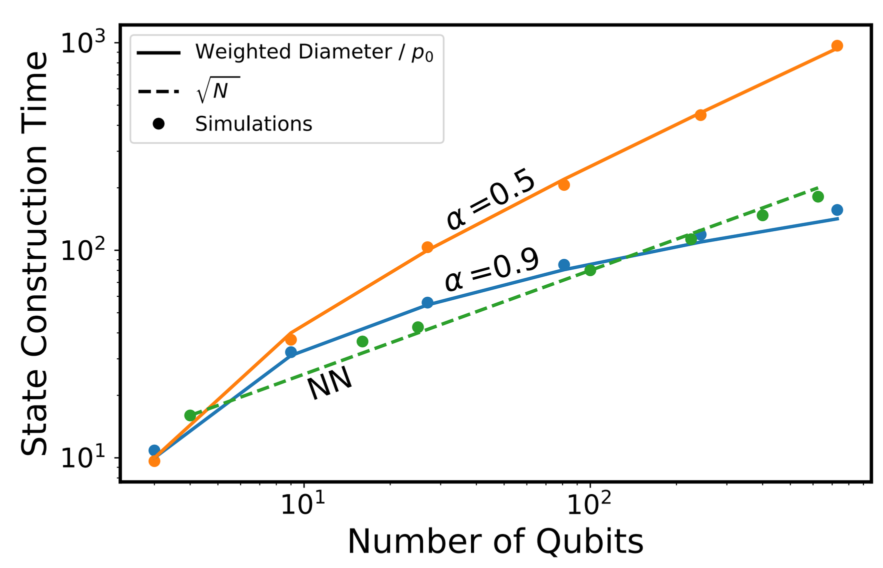
for every eigenvalue κ_j of K.

Entangled State Construction

As a simple benchmark of graph architectures, we calculate the time required to create a GHZ state involving all qubits (one per node) in the graph.
To investigate construction of GHZ states in a heralded entanglement setting, we assume time is discrete and that on each time step, the probability of successful entanglement generation between nodes i and j is w_{ij}. We consider both a 2D nearest-neighbor (NN) grid with $w_{ij} = p_0$ for neighboring qubits, as well as a hierarchical graph where the probability at the ith level of the hierarchy is given by $p_0 \alpha^{i-1}$. For a constant p_0, it is the case that in expectation, the time to construct the GHZ state, t_{GHZ}, is on the order of the weighted diameter,

$$\mathbb{E}[t_{\mathrm{GHZ}}] \sim \mathcal{O}(d_w(G))$$

The figure below shows simulation of t_{GHZ} for the hierarchy $K_3^{\Pi_\alpha i}$ at various α, as well as for a 2D NN grid. In each case, $p_0 = 0.1$. The $\sqrt{N}$ fit shows the scaling of t_{GHZ} for the NN case. With a simple recursive argument, one can show that when $\alpha \geq n^{-1/2}$ (where $n^i = N$ for an i-level hierarchy), the hierarchy beats the NN grid. This is reflected in the figure below.



The hierarchical product formalism introduced here is an experimentally well-motivated and flexible framework that allows easy calculation of relevant graph connectivity properties, spectral properties, and the performance of dynamical processes such as the unitary generated of a large, entangled quantum state on a modular architecture.

Acknowledgments

We thank K. Hyatt, Y.-K. Liu, E. Schoute, D. Wentzlaff, D. Maslov and F. Chong for discussions. This work was supported by ARO MURI, ARL CDQI, NSF QIS, ARO, NSF PFC at JQI, and AFOSR. Z. E. is supported in part by the ARCS Foundation. J. R. G. is supported by the the NIST NRC Research Postdoctoral Associateship Award. A. B. is supported by the QuICS Lanczos Fellowship.

1. C. Monroe and J. Kim, Science 339, 1164 (2013).
2. M. H. Devoret and R. J. Schoelkopf, Science 339, 1169 (2013).
3. A. Acin, J. I. Cirac, and M. Lewenstein, Nat. Phys. 3, 256 (2007).
4. L. Barrire, F. Comellas, C. Dalf, and M. Fiol, Discret. Appl. Math. 157, 36 (2009)
5. M. W. Newman, (2000), and B. Mohar, 2, 871 (1991).