Routing with Reversal

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Part I: Routing

Routing

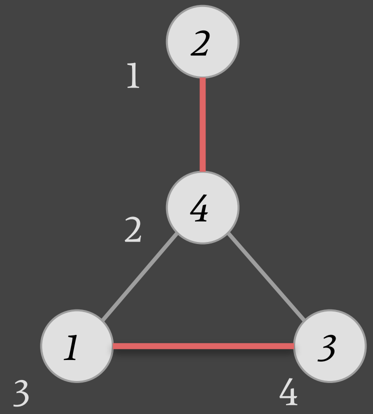
Routing models the problem of information transfer in a network of connected information sources (such as CPUs in a cluster).

Setup: You are given a graph $G = (V, E)$. At $t=0$, every node i has a “packet” with a vertex label $\pi(i)$ on it, where π is a permutation of the nodes.

Allowed moves: swap packets between any two vertices connected by an edge. Each swap consumes 1 time step.

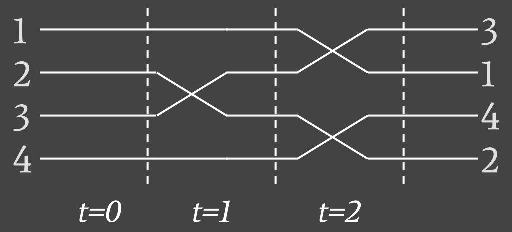
Goal: Route every packet to its given destination in the least possible time.

$\pi = (1243)$



Example

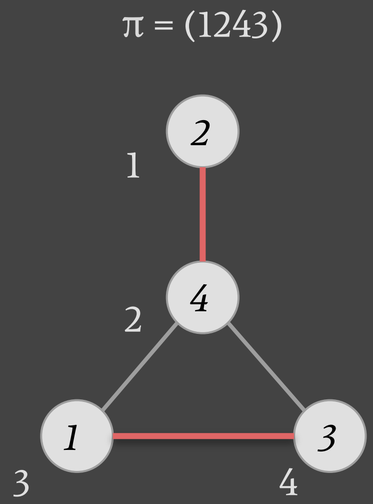
$\pi = (1243) = (23)(12)(34).$



$\mathrm{rt}(G, \pi) := \min_{C(\pi)} \mathrm{time}(C(\pi))$

$\mathrm{rt}(G) := \max_{\pi} \mathrm{rt}(G,\pi)$

So, the routing number of a graph is a functional measure of connectivity that tells us how slow permuting on it can be.



Why do we study routing?

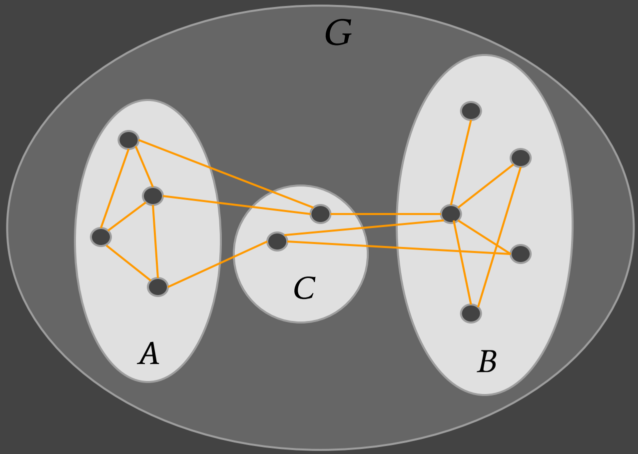
1. Quantum communication will rely on fast (quantum) routing
2. Routing generalizes the task of state transfer
3. Routing can enhance the connectivity of a quantum computer/network.
4. (To show that quantum routing is better than classical routing...)

The routing number is quite hard to pin down - however, bounds exist. For instance:

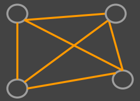
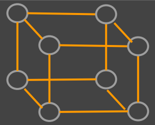
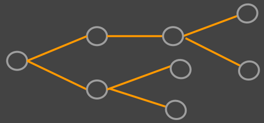
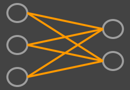
$\mathrm{rt}(G) \geq \mathrm{diam}(G)$

$\mathrm{rt}(G) \leq \mathrm{floor}((3N-1)/2)$

$\mathrm{rt}(G) \geq 2 * \max_C |A|/|C|$ (assuming $|A|\leq|B|$)



Some examples

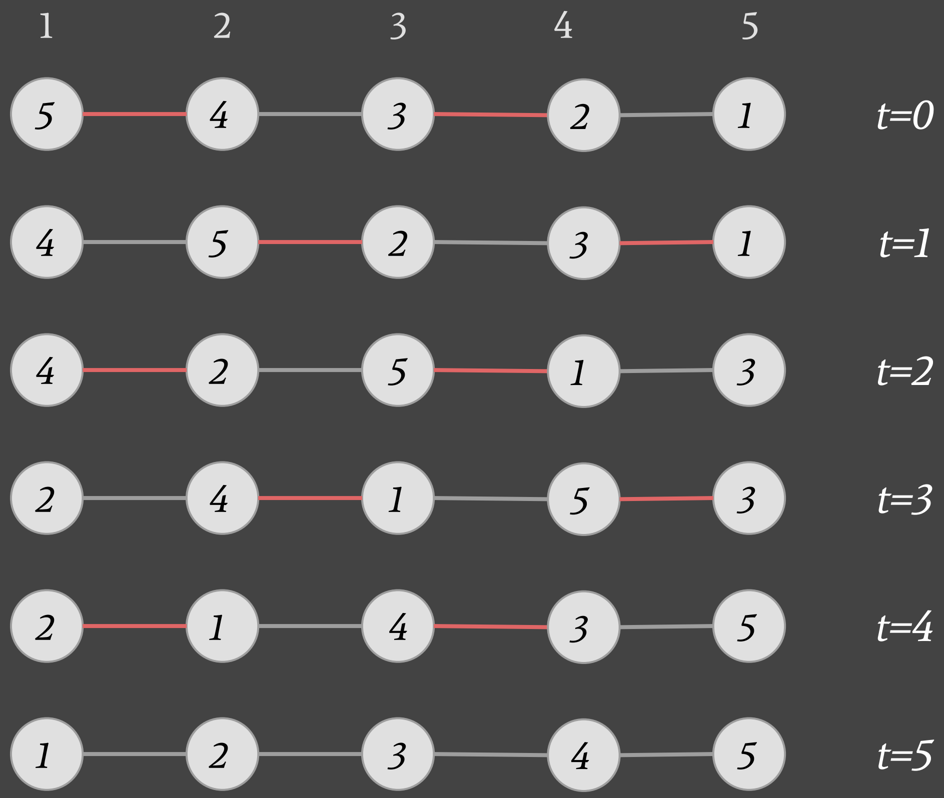
G	Picture	Routing number
Complete, K_N		$\mathrm{rt}(G) = 2$
Path, P_N		$N\text{-}1 \leq \mathrm{rt}(G) \leq N$
Hypercube, Q_n		$n\text{-}1 \leq \mathrm{rt}(G) \leq 2n+2$
Tree		$\mathrm{rt}(G) \leq \mathrm{floor}((3N\text{-}1)/2)$
Complete bipartite, $K_{m,n}$		$\mathrm{rt}(G) \leq \mathrm{floor}((3m)/(2n)) + O(1)$

Routing on the path P_N: odd-even sort

Odd-even sort. Provably routes any permutation in time $t \leq N$.

Algorithm:- Until ordered, do:

- Compare nodes on odd-numbered edges; swap if out of order
- Compare nodes on even-numbered edges; swap if out of order.



Can we do better?

Let's assume that the packet at each node contains a bit of information in addition to a destination vertex.

Suppose we allow other operations on each node and every pair of nodes connected by an edge. Can we route faster than $N-1$?

(Cheating) Just read all the contents of the packet, and write data to destination node.

If contents inaccessible, then time is at least $N-1$. Why?

The ends of the chain are a distance $N-1$ apart. A depth smaller than $N-1$ would imply that they are "causally disconnected" => The diameter bound is inviolable for a state-independent quantum routing algorithm.

Part II: Quantum Routing

Quantum mechanics cheat sheet

Quantum mechanics seeks to describe the universe as a collection of interacting subsystems ("particles") whose internal degrees of freedom are represented by a vector space with an inner product (Hilbert space).

Our "universe" is P_N, where the state of every node lies in a 2D, complex Hilbert space spanned by $|0>$ and $|1>$ (aka a "qubit"). The most general state of a qubit is

$$|\psi> = x|0> + y|1> = \begin{bmatrix} x \\ y \end{bmatrix}$$ where x,y are complex numbers s.t. $|x|^2 + |y|^2 = 1.$

The state of the full path lives in a space that is a tensor product of the N qubit Hilbert spaces. Therefore it is a state of the form

$$|\Psi> = \sum x_b |b_1 b_2 b_3 .. b_N>$$ where $b_i \in \{0,1\}$ and x_b are complex numbers s.t. $\sum |x_b|^2 = 1.$

Quantum mechanics cheat sheet

The state evolves in time via the Schrödinger equation. Since norm must be preserved, the evolution is a rotation in Hilbert space. A Hermitian matrix H describes the evolution:

$$d|\Psi\rangle/dt = -iH|\Psi\rangle \Rightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

The full state is inaccessible to an observer. In particular, the state *cannot be copied* (quantum no-cloning theorem).

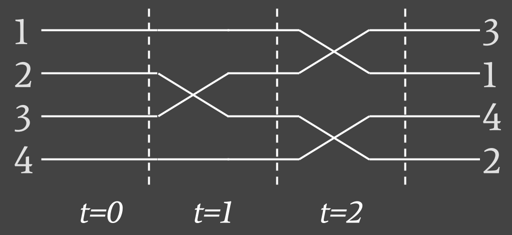
In order to gain information about the state, a measurement must be performed. A simple example of a measurement is to check if a given node is in state $|0\rangle$ or $|1\rangle$. Once measured, the qubit in that node collapses to the observed outcome.

Quantum routing

Now, every node contains a qubit that must be transported to its destination. Without peeking!

A classical swap-based routing protocol directly gives a quantum routing protocol:

$$\text{Swap} \dashrightarrow \text{SWAP gate} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} |00> \\ |01> \\ |10> \\ |11> \end{array} \\ \begin{array}{cccc} |00> & |01> & |10> & |11> \end{array} & \end{array} = exp(-iH), \text{ where } H = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \pi/2 & 0 \\ 0 & \pi/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} |00> \\ |01> \\ |10> \\ |11> \end{array} \\ \begin{array}{cccc} |00> & |01> & |10> & |11> \end{array} & \end{array}$$



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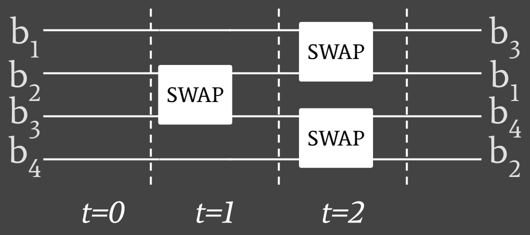


Hamiltonian routing

Gates are limited by their “discreteness”. Can we do better using continuous-time Hamiltonian evolution?

The Hamiltonian describes how the qubits interact with one another. In a graph with limited connectivity, the interactions are constrained to act on nodes connected by an edge.

Can we engineer a Hamiltonian that implements routing? In other words, can we come up with a Hamiltonian H_π *so that*

$$e^{-iHt}|\Psi> = e^{-iHt}\sum x_b|b_1b_2b_3..b_N> \quad = \quad \sum x_b|b_{\pi(1)}\, b_{\pi(2)}\, b_{\pi(3)} ...b_{\pi(N)} >$$

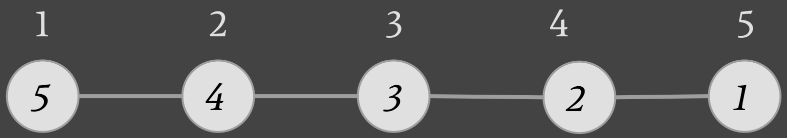
for a specific permutation π ? For all permutations ? Never ?

Yes! A “reversal” can be implemented faster than classically using an engineered Hamiltonian.

Part II: Routing with Reversal

Fast reversal

A reversal R is a permutation that “flips” the chain about the center:



Classically, the best routing time to implement R is at least *N-1* (diameter bound). Surprisingly, there is a Hamiltonian, state-independent protocol that achieves this in time (N+1)/3!

Goals:

(For physicists) Can we find protocols for other permutations on the path? On general graphs?

(For math/CS folks) Can we use fast reversal as a primitive to speed up other permutations?

The goals of this project

New model: Our allowed operations now include all permutations of the form

$R(i,j) = \prod_{k=0}^{floor((j-i)/2)} (i+k\ j-k)$ that take time $t(i,j) := (|j-i|+2)/3$.

The "routing with reversal" number $rrt(G)$ is then defined analogously to $rt(G)$.

We wish to answer a non-zero number of the following questions:

1. What is a lower bound on $rrt(G)$? $rrt(G,\pi)$?
2. Is it tight? I.e., what is $rrt(G)$? $rrt(G,\pi)$?
3. Can we construct an algorithm whose runtime is optimal?
4. Can we construct an algorithm that provably beats odd-even sort?
5. Is deciding whether routing with reversal beats odd-even sort NP-hard?
6. ....

Thanks!

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